Randomized Unscented Transform in State Estimation of non-Gaussian Systems: Algorithms and Performance

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***Abstract*—The paper deals with state estimation of nonlinear non-Gaussian systems with a special focus on the Gaussian sum filters. To achieve a higher estimate quality, state and measurement predictive moments appearing in the filters are computed by the randomized unscented transform, which provides asymptotically exact estimates of the moments. The use of the Gaussian sum filter employing the randomized unscented transform is introduced and the proposed algorithm is illustrated in a numerical example. The analysis of the numerical example involves a comparison of several filters using a number of performance metrics both absolute and relative, assessing the point estimate quality, the estimate error quality, and the density estimate quality.**

Keywords: state estimation, nonlinear filtering, non-Gaussian.

I. Introduction

State estimation of nonlinear non-Gaussian discrete-time stochastic dynamic systems is of utter importance in many areas such as target tracking [1], [2], navigation, signal processing, fault detection, and adaptive and optimal control problems.

The Bayesian framework is a suitable tool for the state estimation problem as it enables a non-Gaussianity analysis with respect to the description of random quantities. A general solution to recursive state estimation problems within the Bayesian framework, is given by the Bayesian recursive relations (BRR's). They produce probability density functions (pdf's) of the state conditioned by the measurements. The pdf's represent a full stochastic description of the state, which itself is unmeasurable.

The closed-form solution to the BRR's is available only for a few special cases, e.g., for a linear Gaussian system [3], where the BRR's solution corresponds to the Kalman filter. When the measurement model or the state transition model is nonlinear, an approximate method must be applied. These approximate methods can be divided into two groups with respect to the validity of the resulting estimates [4]. The first group of methods provides results with validity within a neighborhood of the point estimate only and thus they are called *local methods*. The second group of methods provides results valid within almost the whole state space and are called *global methods*.

The local methods provide approximate conditional mean and associated covariance matrix of the estimate error. Popular approximations to nonlinear functions used by the methods are (i) the Taylor series expansion of the nonlinear functions in the system description leading to the extended Kalman filter (EKF) or the second-order extended Kalman filter [5], (ii) the polynomial linearization of the nonlinear function by a first or second order polynomial interpolation [6], [7] leading to the divided difference filters (DDF's), (iii) the stochastic linearization [8] approximating a random variable by a set of points, which are transformed through nonlinear functions and leading to the unscented Kalman filter (UKF) [9]–[11], and (iv) calculating approximate moments of a random variable by means of the Gauss-Hermite quadrature [6] or cubature integration rules and leading to quadrature or cubature filters [12]. In [13] a randomized unscented Kalman filter has been proposed (RUKF) based on a degree 3 stochastic integration rule (SIR3), which is combination of the cubature rule and the Monte Carlo (MC) method.

Due to the approximation of the state estimate conditional pdf by the first two moments only, the local methods are not very practical for non-Gaussian problems. Here, the global estimation methods, providing an approximation of the full conditional pdf, achieve a good quality performance. However, note that the improved performance is usually paid by higher computational costs. There are three main approaches to the global filtering method design: (i) the analytical approach based on Gaussian sum approximation of pdf's [14], [15] and using approximation techniques of the local methods, (ii) the numerical approach using point-mass approximation of the conditional state pdf's and solving the integrals in the BRR's numerically [16], [17], and (iii) the simulation approach taking advantage of the BRR's solution by the MC methods and approximating the conditional state pdf by an empirical representation [18], [19].

The analytical global methods are based on a multiple

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14. ABSTRACT
The paper deals with state estimation of nonlinear non-Gaussian systems with a special focus on the Gaussian sum filters. To achieve a higher estimate quality, state and measurement predictive moments appearing in the filters are computed by the randomized unscented transform, which provides asymptotically exact estimates of the moments. The use of the Gaussian sum filter employing the randomized unscented transform is introduced and the proposed algorithm is illustrated in a numerical example. The analysis of the numerical example involves a comparison of several filters using a number of performance metrics both absolute and relative, assessing the point estimate quality, the estimate error quality, and the density estimate quality.

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application of local methods, e.g. the Gaussian Sum Filter (GSF) consists of a bank of EKF's, or the Sigma Point Gaussian Sum Filter [20] using a bank of DDF's or UKF's. In problems involving multiple mode pdf's, the analytical global methods provide a high estimate quality as they are able to follow the modes with individual local methods.

The goal of the paper is to propose a filter based on Gaussian sum representation of pdf's and utilizing the randomized unscented transform (RUT) which has been used in [13] as a basis for designing the RUKF. The RUT will be used to compute statistics of a transformed random variable. Comparing to the unscented transform (UT), appearing in the UKF, the RUT offers asymptotically exact estimates of the statistics. The RUT gives a flexible and computationally efficient method for computing posterior moments when measurement and/or kinematic models are nonlinear.

The paper is organized as follows: System specification and nonlinear non-Gaussian state estimation by means of a generic Gaussian-sum based global filter are introduced in Section II. The RUT is discussed in Section III. Section IV deals with the algorithm of the proposed filter with a special focus on implementation details. In Section V, the proposed filter will be applied to a numerical example for performance evaluation and comparison, and concluding remarks are drawn in Section VI.

II. System Specification and Generic Gaussian-sum Based Global Filter

Let the discrete-time nonlinear stochastic system be considered in the following form

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k) + \mathbf{w}_k, k = 0, 1, 2, \ldots, \tag{1}$$

$$\mathbf{z}_k = \mathbf{h}_k(\mathbf{x}_k) + \mathbf{v}_k, k = 0, 1, 2, \ldots, \tag{2}$$

where the vectors $\mathbf{x}_k \in \mathbb{R}^{n_x}$ and $\mathbf{z}_k \in \mathbb{R}^{n_z}$ represent the unmeasurable state of the system and measurement at time instant k, respectively, $\mathbf{f}_k : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ and $\mathbf{h}_k : \mathbb{R}^{n_x} \to \mathbb{R}^{n_z}$ are known vector functions, and $\mathbf{w}_k \in \mathbb{R}^{n_x}$, $\mathbf{v}_k \in \mathbb{R}^{n_z}$ are the state and measurement white noises, which are supposed to be mutually independent. The pdf's of the noises $p(\mathbf{w}_k)$ and $p(\mathbf{v}_k)$ are assumed to be known. The initial state $\mathbf{x}_0$ is independent of the noises and its pdf is supposed to be known.

The general solution to the estimation problem (i.e. finding $\mathbf{x}_k$ based on knowledge of $\mathbf{z}^k \triangleq [\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_k]$) is given by the BRR's computing pdf's of the state conditioned by the measurements [3]. These pdf's provide a full description of the estimated state. The BRR's are assumed in the following form

$$p(\mathbf{x}_k|\mathbf{z}^k) = \frac{p(\mathbf{x}_k|\mathbf{z}^{k-1})p(\mathbf{z}_k|\mathbf{x}_k)}{p(\mathbf{z}_k|\mathbf{z}^{k-1})}, \tag{3}$$

$$p(\mathbf{x}_{k+1}|\mathbf{z}^k) = \int p(\mathbf{x}_{k+1}|\mathbf{x}_k)p(\mathbf{x}_k|\mathbf{z}^k)d\mathbf{x}_k, \tag{4}$$

where $p(\mathbf{x}_k|\mathbf{z}^k)$ is the filtering pdf, $p(\mathbf{x}_{k+1}|\mathbf{z}^k)$ is the one-step ahead predictive pdf, $p(\mathbf{x}_{k+1}|\mathbf{x}_k)$ and $p(\mathbf{z}_k|\mathbf{x}_k)$ are the state transition pdf obtained from (1) and the measurement pdf obtained from (2), respectively, and $p(\mathbf{z}_k|\mathbf{z}^{k-1}) = \int p(\mathbf{x}_k|\mathbf{z}^{k-1})p(\mathbf{z}_k|\mathbf{x}_k)d\mathbf{x}_k$. The closed form solution to the BRR's is available only for a few special cases [3]. In other cases it is necessary to apply an approximation in the BRR's solution.

The Gaussian sum-based global filter requires specification of the noises and initial state pdf's in the form of Gaussian sums as follows:

$$p(\mathbf{x}_0) = \sum_{i=1}^{N_0} \alpha_0^{(i)} \mathcal{N}\{\mathbf{x}_0; \hat{\mathbf{x}}_0^{(i)}, \mathbf{P}_0^{(i)}\}, \tag{5}$$

$$p(\mathbf{w}_k) = \sum_{i=1}^{q_k} \beta_k^{(i)} \mathcal{N}\{\mathbf{w}_k; \hat{\mathbf{w}}_k^{(i)}, \mathbf{Q}_k^{(i)}\}, \tag{6}$$

$$p(\mathbf{v}_k) = \sum_{i=1}^{r_k} \gamma_k^{(i)} \mathcal{N}\{\mathbf{v}_k; \hat{\mathbf{v}}_k^{(i)}, \mathbf{R}_k^{(i)}\}. \tag{7}$$

where $\mathcal{N}\{\mathbf{y}; \hat{\mathbf{y}}, \mathbf{P}_y\}$ denotes Gaussian distribution of a random variable $\mathbf{y}$ parametrized by its mean $\hat{\mathbf{y}}$ and covariance matrix $\mathbf{P}_y$. The parameters $\alpha_0^{(i)}$, $\beta_k^{(i)}$, and $\gamma_k^{(i)}$ are positive weights of particular Gaussian terms with their sum being equal to 1. All the parameters (i.e. the weights, means and covariance matrices) are assumed to be known.

If any of the noises or the initial state has a distribution different from Gaussian sum distribution, its Gaussian sum approximation has to be found (e.g. using the expectation-maximization (EM) method [21]). It has been proved that the approximation by a Gaussian sum can be arbitrarily accurate [3]. Considering the noises given by the Gaussian sum pdf (5–7), the following algorithm of a global filter provides a generic solution to the estimation problem.

Algorithm 1: Generic Gaussian Sum Global Filter

Step 1: (*initialization*) Set the time instant $k = 0$ and define a priori initial condition $p(\mathbf{x}_0|\mathbf{z}^{-1}) = p(\mathbf{x}_0)$ as a sum of $N_{0|-1} = N_0$ Gaussian terms.

Step 2: (*filtering*) The filtering pdf is approximated by

$$p(\mathbf{x}_k|\mathbf{z}^k) \approx \sum_{i=1}^{N_{k|k}} \alpha_{k|k}^{(i)} \mathcal{N}\{\mathbf{x}_k; \hat{\mathbf{x}}_{k|k}^{(i)}, \mathbf{P}_{k|k}^{(i)}\}, \tag{8}$$

where $N_{k|k} = N_{k|k-1} \cdot r_k$. The filtering mean $\hat{\mathbf{x}}_{k|k}^{(i)}$ and the covariance matrix $\mathbf{P}_{k|k}^{(i)}$ of the i-th term $\mathcal{N}\{\mathbf{x}_k; \hat{\mathbf{x}}_{k|k}^{(i)}, \mathbf{P}_{k|k}^{(i)}\}$ are computed using

$$\hat{\mathbf{x}}_{k|k}^{(i)} = \hat{\mathbf{x}}_{k|k-1}^{(j)} + \mathbf{K}_{k|k}^{(i)}(\mathbf{z}_k - \hat{\mathbf{z}}_{k|k-1}^{(i)}), \tag{9}$$

$$\mathbf{P}_{k|k}^{(i)} = \mathbf{P}_{k|k-1}^{(j)} - \mathbf{K}_{k|k}^{(i)}\mathbf{P}_{z,k|k-1}^{(i)}(\mathbf{K}_{k|k}^{(i)})^{\mathrm{T}}, \tag{10}$$

where $\mathbf{K}_{k|k}^{(i)} = \mathbf{P}_{xz,k|k-1}^{(i)} \left(\mathbf{P}_{z,k|k-1}^{(i)}\right)^{-1}$ and

$$\hat{\mathbf{z}}_{k|k-1}^{(i)} = \mathsf{E}_{k|k-1}^{(j)}[\mathbf{h}(\mathbf{x}_k)] + \hat{\mathbf{v}}_k^{(l)}, \tag{11}$$

$$\mathbf{P}_{z,k|k-1}^{(i)} = \mathsf{cov}_{k|k-1}^{(j)}[\mathbf{h}(\mathbf{x}_k)] + \mathbf{R}_k^{(l)}, \tag{12}$$

$$\mathbf{P}_{xz,k|k-1}^{(i)} = \mathsf{E}_{k|k-1}^{(j)}[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k-1}^{(j)})(\mathbf{h}_k(\mathbf{x}_k) - \hat{\mathbf{z}}_{k|k-1}^{(i)})^T], \tag{13}$$

where $\mathsf{E}^{(j)}_{k|k-1}[\cdot]$ is a shorthand notation for

$$\mathsf{E}_{\mathcal{N}\{\mathbf{x}_k;\hat{\mathbf{x}}^{(j)}_{k|k-1},\mathbf{P}^{(j)}_{k|k-1}\}}[\cdot] = \int [\cdot]\mathcal{N}\{\mathbf{x}_k;\hat{\mathbf{x}}^{(j)}_{k|k-1},\mathbf{P}^{(j)}_{k|k-1}\}\mathrm{d}\mathbf{x}_k$$

and $\mathsf{cov}^{(j)}_{k|k-1}[\cdot]$ for $\mathsf{cov}_{\mathcal{N}\{\mathbf{x}_k;\hat{\mathbf{x}}^{(j)}_{k|k-1},\mathbf{P}^{(j)}_{k|k-1}\}}[\cdot]$.

The likelihood $\zeta^{(i)}_k$ of the ith term with respect to the known measurement $\mathbf{z}_k$ is given by

$$\zeta^{(i)}_k \approx \mathcal{N}\{\mathbf{z}_k;\hat{\mathbf{z}}^{(i)}_{k|k-1},\mathbf{P}^{(i)}_{z,k|k-1}\}. \quad (14)$$

The indices j and l are given by

$$j = i - \lfloor\frac{i-1}{N_{k|k-1}}\rfloor N_{k|k-1}, \quad j = 1,\dots,N_{k|k-1}, \quad (15)$$

$$l = 1 + \lfloor\frac{i-1}{N_{k|k-1}}\rfloor, \quad l = 1,\dots,r_k \quad (16)$$

and $i = 1,\dots,N_{k|k}$. The symbol $\lfloor x\rfloor$ denotes the floor function, i.e. the largest integer less than or equal to x. The filtering weight $\alpha^{(i)}_{k|k}$ of each term is given by

$$\alpha^{(i)}_{k|k} = \frac{\alpha^{(j)}_{k|k-1}\gamma^{(l)}_k\zeta^{(i)}_k}{\sum_{j=1}^{N_{k|k-1}}\sum_{l=1}^{r_k}\alpha^{(j)}_{k|k-1}\gamma^{(l)}_k\zeta^{(N_{k|k-1}(j-1)+l)}_k}. \quad (17)$$

Step 3: (*global point estimate*) The global filtering mean and covariance matrix can be obtained by the following relations

$$\hat{\mathbf{x}}_{k|k} = \mathsf{E}[\mathbf{x}_k|\mathbf{z}^k] = \sum_{i=1}^{N_{k|k}}\alpha^{(i)}_{k|k}\hat{\mathbf{x}}^{(i)}_{k|k}, \quad (18)$$

$$\begin{aligned}\mathbf{P}_{k|k} &= \mathsf{cov}[\mathbf{x}_k|\mathbf{z}^k]\\ &= \sum_{i=1}^{N_{k|k}}\alpha^{(i)}_{k|k}\left[\mathbf{P}^{(i)}_{k|k} + (\hat{\mathbf{x}}_{k|k}-\hat{\mathbf{x}}^{(i)}_{k|k})(\hat{\mathbf{x}}_{k|k}-\hat{\mathbf{x}}^{(i)}_{k|k})^T\right]. \quad (19)\end{aligned}$$

Similarly, other point estimates (e.g. a mode or median) can be obtained from the global state estimate (8).

Step 4: (*reduction*) Generally, the state or measurement noise with the Gaussian sum distribution causes an exponential growth of the number of Gaussian terms in the sum (8), which must be reduced to keep computational costs reasonable [22], [23].

Step 5: (*prediction*) The predictive pdf is approximated by

$$p(\mathbf{x}_{k+1}|\mathbf{z}^k) \approx \sum_{i=1}^{N_{k+1|k}}\alpha^{(i)}_{k+1|k}\mathcal{N}\{\mathbf{x}_{k+1};\hat{\mathbf{x}}^{(i)}_{k+1|k},\mathbf{P}^{(i)}_{k+1|k}\}, \quad (20)$$

where $N_{k+1|k} = N_{k|k}\cdot q_k$ and $\alpha^{(i)}_{k+1|k} = \alpha^{(j)}_{k|k}\beta^{(l)}_k$ The particular predictive mean $\hat{\mathbf{x}}^{(i)}_{k+1|k}$ and covariance matrix $\mathbf{P}^{(i)}_{k+1|k}$ are computed by

$$\hat{\mathbf{x}}^{(i)}_{k+1|k} = \mathsf{E}^{(j)}_{k|k-1}[\mathbf{f}_k(\mathbf{x}_k)] + \hat{\mathbf{w}}^{(l)}_k, \quad (21)$$

$$\mathbf{P}^{(i)}_{k+1|k} = \mathsf{cov}^{(j)}_{k|k-1}[\mathbf{f}_k(\mathbf{x}_k)] + \mathbf{Q}^{(l)}_k. \quad (22)$$

Again, the indices j and l are given by

$$j = i - \lfloor\frac{i-1}{N_{k|k}}\rfloor N_{k|k}, \quad j = 1,\dots,N_{k|k}, \quad (23)$$

$$l = 1 + \lfloor\frac{i-1}{N_{k|k}}\rfloor, \quad l = 1,\dots,q_k \quad (24)$$

and $i = 1,\dots,N_{k+1|k}$.

Let $k = k+1$, Then go to **Step 2**.

III. Randomized Unscented Transform

The RUKF proposed in [13] employs the RUT as a mean to calculate approximate predictive statistics (state means and covariance matrices) of the state and measurement. The RUT is a special case of SIR3 proposed in [24], [25]. The SIR3 aims at evaluating an integral of the form

$$\boldsymbol{\mu} = \int_{\mathbb{R}^{n_x}}\boldsymbol{\varphi}(\mathbf{x})(2\pi)^{-n_x/2}|\mathbf{P}|^{-1/2}\mathbf{e}^{-\frac{1}{2}(\mathbf{x}-\hat{\mathbf{x}})^T\mathbf{P}^{-1}(\mathbf{x}-\hat{\mathbf{x}})}\mathrm{d}\mathbf{x}, \quad (25)$$

where $\boldsymbol{\varphi}(\cdot)$ is an arbitrary function. Note that relation (25) can be interpreted as computation of the expected value of the function $\boldsymbol{\varphi}(\mathbf{x})$ where $\mathbf{x}$ is a random variable with $p(\mathbf{x}) = \mathcal{N}\{\mathbf{x};\hat{\mathbf{x}},\mathbf{P}\}$, i.e.

$$\boldsymbol{\mu} = E_{\mathcal{N}\{\mathbf{x};\hat{\mathbf{x}},\mathbf{P}\}}[\boldsymbol{\varphi}(\mathbf{x})]. \quad (26)$$

The SIR3 proposed in [24] for a solution to (25) is given by the following algorithm:

Algorithm 2: Degree 3 stochastic integration rule

Step 1: Choose an error tolerance ε and a maximum number of iterations N_{max}.

Step 2: Initialize the number of iterations $N = 0$, initial value of the integral $\hat{\boldsymbol{\mu}} = \mathbf{0}_{n_x\times 1}$, and initial square-error of the integral estimate $\mathbf{V} = \mathbf{0}_{n_x\times n_x}$ and calculate a square root $\sqrt{\mathbf{P}_x}$ of $\mathbf{P}_x$ such that $\mathbf{P}_x = \sqrt{\mathbf{P}_x}\sqrt{\mathbf{P}_x}^{\mathrm{T}}$. Note that $\mathbf{0}_{a\times b}$ denotes a $a\times b$ matrix of zeros.

Step 3: Repeat (until $N = N_{max}$ or $\|\mathbf{V}\| < \varepsilon$) the following loop:

a) Set $N = N+1$.

b) Generate a uniformly random orthogonal matrix $\mathbf{U}$ of dimension $n_x\times n_x$ and generate a random number ρ from chi-squared distribution $\rho \sim \mathrm{Chi}(n_x+2)$.

c) Compute a set of points $\boldsymbol{\chi}_i$ and appropriate weights ω_i according to

$$\begin{aligned}\boldsymbol{\chi}_0 &= \hat{\mathbf{x}}, & \omega_0 &= 1-\tfrac{n_x}{\rho^2},\\ \boldsymbol{\chi}_i &= \hat{\mathbf{x}} - \rho\mathbf{U}(\sqrt{\mathbf{P}_x})_i, & \omega_i &= \tfrac{1}{2\rho^2},\\ \boldsymbol{\chi}_{n_x+i} &= \hat{\mathbf{x}} + \rho\mathbf{U}(\sqrt{\mathbf{P}_x})_i, & \omega_{n_x+i} &= \omega_i,\end{aligned}$$

where $i = 1,2,\dots,n_x$ and the term $\left(\sqrt{\mathbf{P}_x}\right)_i$ represents the i-th column of the matrix $\sqrt{\mathbf{P}_x}$.

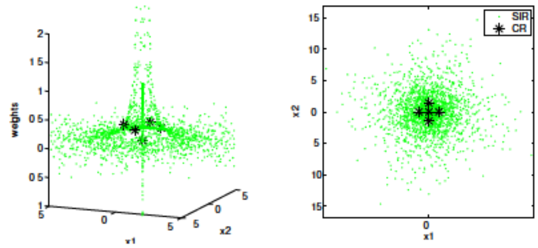

Figure 1. Comparison of points and weights generated by the CR and SIR3 for $\mathbb{R}^2$ state space: left - comparison of the weights, right - comparison of the points placement (dots - SIR3, stars - CR).

d) Compute the value $\mathbf{S}$ of the integral at current iteration

$$\mathbf{S} = \sum_{i=0}^{2n_x} \omega_i \varphi(\chi_i),$$

and use it to update the approximate value $\hat{\mu}$ of the integral and square-error of the integral estimate $\mathbf{V}$ as

$$\begin{aligned}\mathbf{D} &= (\mathbf{S} - \hat{\mu})/N,\\ \hat{\mu} &= \hat{\mu} + \mathbf{D},\\ \mathbf{V} &= (N-2)\mathbf{V}/N + \mathbf{D}\mathbf{D}^T.\end{aligned}$$

The SIR3 can be thought of as a blend of a cubature rule (CR) and the MC method. The SIR3 consists of the spherical integration rule, which generates a set of $2n_x$ points located on an unit n_x-sphere, and the radial integration rule, which governs spread of the points within the $\mathbb{R}^{n_x}$.

In Figure 1, a comparison of the points and weights generated by the CR and the points and weights generated by the SIR's is shown. Contrary to the CR, the SIR3 provides asymptotically exact estimates of integral [24], i.e. $\mathsf{E}[\hat{\mu}] = \mu$.

Note that the points generated by the CR are identical with the points generated by the UT with scaling parameter $\kappa = 0$. In [13], it was shown that the location of points generated by the SIR3 when choosing a trivial orthogonal matrix $\mathbf{U} = \mathbf{I}$ and the parameter ρ as $\rho^2 = \kappa + n_x$ matches the location of points generated by the UT which uses the scaling parameter κ. This fact lead to coining the term *randomized UT* in [13] for the SIR3 used to estimate moments of a random variable $\mathbf{y} \in \mathbb{R}^{n_y}$ related to a random variable $\mathbf{x} \in \mathbb{R}^{n_x}$ that is given by its mean $\hat{\mathbf{x}}$ and covariance matrix $\mathbf{P}_x$ through a known nonlinear function $\mathbf{y} = \mathbf{g}(\mathbf{x}) = [g_1(\mathbf{x}), \ldots, g_{n_y}(\mathbf{x})]^T$. The RUT estimates the mean $\mathsf{E}[\mathbf{y}] = \mathsf{E}[\mathbf{g}(\mathbf{x})]$ and the covariance matrix $\mathsf{cov}[\mathbf{y}] = \mathsf{E}[(\mathbf{y}-\hat{\mathbf{y}})(\mathbf{y}-\hat{\mathbf{y}})^T]$ of $\mathbf{y}$, and the cross-covariance matrix $\mathsf{E}[(\mathbf{x}-\hat{\mathbf{x}})(\mathbf{y}-\hat{\mathbf{y}})^T]$.

IV. Gaussian Sum Global Filter with Randomized Unscented Transform

The RUT introduced in the previous section will be used to calculate the predictive measurement moments (11–13) and the predictive state moments (21–22) of the Gaussian sum global filter described in Algorithm 1. Due to paper size limitations, only measurement prediction moments will be discussed in this section. Calculation of the state prediction moments is analogous.

The simplest application of the RUT in computation of the predictive measurement moments in Algorithm 1 would comprise a SIR3 algorithm iterative run for each of $N_{k|k-1}$ means $\mathsf{E}^{(j)}_{k|k-1}[\mathbf{h}_k(\mathbf{x}_k)]$, $N_{k|k-1}$ covariance matrices $\mathsf{cov}^{(j)}_{k|k-1}[\mathbf{h}_k(\mathbf{x}_k)]$ and $N_{k|k-1}$ cross-covariance matrices $\mathbf{P}^{(i)}_{xz,k|k-1}$. Naturally, a large number of random parameters ρ and random orthogonal matrices $\mathbf{Q}$ would have to be drawn which would lead to a substantial increase in computational costs. To increase the timeliness in the presence of high computational costs, it is possible to compute all the moments in parallel. This technique will be adopted in the following algorithm.

Additionally, to reduce the costs even further, the algorithm will compute second raw moments $\mathsf{E}^{(j)}_{k|k-1}[\mathbf{h}_k(\mathbf{x}_k)\mathbf{h}_k(\mathbf{x}_k)^{\mathrm{T}}]$ and $\mathsf{E}^{(j)}_{k|k-1}[\mathbf{x}_k\mathbf{h}_k(\mathbf{x}_k)^{\mathrm{T}}]$ instead of the covariance and cross-covariance matrices $\mathsf{cov}^{(j)}_{k|k-1}[\mathbf{h}_k(\mathbf{x}_k)]$ and $\mathbf{P}^{(i)}_{xz,k|k-1}$, respectively. As the last step of the algorithm, the covariance and cross-covariance matrices will be obtained by a trivial combination of the means and second raw moments according to $\mathsf{cov}(\mathbf{y}) = \mathsf{E}[\mathbf{y}\mathbf{y}^{\mathrm{T}}] - \mathsf{E}[\mathbf{y}][\mathsf{E}[\mathbf{y}]]^{\mathrm{T}}$. This enables to run a parallel computation of all terms involved in the Gaussian sum (8).

For notational simplicity, the algorithm will consider a fixed number of iterations. Utilization of the error tolerance ε will be discussed later. Also for the sake of clarity, the time indices will be dropped in the algorithm. Note that $\hat{\mu}$ and $\mathbf{S}$ in Algorithm 2 corresponds to $\hat{\mathsf{E}}^{(j)}[\cdot]$ and $\mathsf{E}^{(j),N}[\cdot]$, respectively in Algorithm 3. .

Algorithm 3: RUT for Gaussian sum filter

Step 1: Choose the total number of iterations N_{total} and initialize the number of iterations as $N = 0$. Set the initial values of the measurement statistics to be calculated:

- $\hat{\mathsf{E}}^{(j)}[\mathbf{h}(\mathbf{x})] = \mathbf{0}_{n_z \times 1}$,
- $\hat{\mathsf{E}}^{(j)}[\mathbf{h}(\mathbf{x})\mathbf{h}(\mathbf{x})^{\mathrm{T}}] = \mathbf{0}_{n_z \times n_z}$,
- $\hat{\mathsf{E}}^{(j)}[\mathbf{x}\mathbf{h}(\mathbf{x})^{\mathrm{T}}] = \mathbf{0}_{n_x \times n_z}$.

Calculate a square root $\sqrt{\mathbf{P}^{(j)}}$ of the state predictive covariance matrices $\mathbf{P}^{(j)}$.

Step 2: Repeat until $N = N_{total}$, the following loop:

a) Set $N = N + 1$.

b) Generate a uniformly random orthogonal matrix $\mathbf{U}$ of dimension $n_x \times n_x$ and generate a random number ρ according to $\rho \sim \mathrm{Chi}(n_x + 2)$.

c) For each of the predictive terms $\mathcal{N}\{\mathbf{x}; \hat{\mathbf{x}}^{(j)}, \mathbf{P}^{(j)}\}$ do the following

   c1) Compute a set of points χ_i and appropriate weights ω_i according to

$$\chi_0^{(j)} = \hat{\mathbf{x}}^{(j)},\ \omega_0 = 1 - \frac{n_x}{\rho^2},$$
$$\chi_i^{(j)} = \hat{\mathbf{x}}^{(j)} - \rho \mathbf{U}(\sqrt{\mathbf{P}^{(j)}})_i,\ \omega_i = \frac{1}{2\rho^2},$$
$$\chi_{n_x+i}^{(j)} = \hat{\mathbf{x}}^{(j)} + \rho \mathbf{U}(\sqrt{\mathbf{P}^{(j)}})_i,\ \omega_{n_x+i} = \omega_i,$$

where $i = 1, 2, \ldots, n_x$.

c2) Compute the values $\mathsf{E}^{(j),N}[\mathbf{h}(\mathbf{x})]$, $\mathsf{E}^{(j),N}[\mathbf{h}(\mathbf{x})\mathbf{h}(\mathbf{x})^{\mathrm{T}}]$, and $\mathsf{E}^{(j),N}[\mathbf{x}\mathbf{h}(\mathbf{x})^{\mathrm{T}}]$ of the statistics at current iteration

$$\mathsf{E}^{(j),N}[\mathbf{h}(\mathbf{x})] = \sum_{i=0}^{2n_x} \omega_i \mathbf{h}(\chi_i^{(j)}),$$
$$\mathsf{E}^{(j),N}[\mathbf{h}(\mathbf{x})\mathbf{h}(\mathbf{x})^{\mathrm{T}}] = \sum_{i=0}^{2n_x} \omega_i \mathbf{h}(\chi_i^{(j)})\mathbf{h}(\chi_i^{(j)})^{\mathrm{T}},$$
$$\mathsf{E}^{(j),N}[\mathbf{x}\mathbf{h}(\mathbf{x})^{\mathrm{T}}] = \sum_{i=0}^{2n_x} \omega_i \chi_i^{(j)} \mathbf{h}(\chi_i^{(j)})^{\mathrm{T}}.$$

Step 3: Calculate values of the statistics as

$$\hat{\mathsf{E}}^{(j)}[\mathbf{h}(\mathbf{x})] = \frac{1}{N} \sum_{N=1}^{N_{total}} \mathsf{E}^{(j),N}[\mathbf{h}(\mathbf{x})],$$
$$\hat{\mathsf{E}}^{(j)}[\mathbf{h}(\mathbf{x})\mathbf{h}(\mathbf{x})^{\mathrm{T}}] = \frac{1}{N} \sum_{N=1}^{N_{total}} \mathsf{E}^{(j),N}[\mathbf{h}(\mathbf{x})\mathbf{h}(\mathbf{x})^{\mathrm{T}}],$$
$$\hat{\mathsf{E}}^{(j)}[\mathbf{x}\mathbf{h}(\mathbf{x})^{\mathrm{T}}] = \frac{1}{N} \sum_{N=1}^{N_{total}} \mathsf{E}^{(j),N}[\mathbf{x}\mathbf{h}(\mathbf{x})^{\mathrm{T}}].$$

Step 4: Find the covariance and cross-covariance matrices as

$$\hat{\mathsf{cov}}^{(j)}[\mathbf{h}(\mathbf{x})] = \hat{\mathsf{E}}^{(j)}[\mathbf{h}(\mathbf{x})\mathbf{h}(\mathbf{x})^{\mathrm{T}}] - \hat{\mathsf{E}}^{(j)}[\mathbf{h}(\mathbf{x})]\hat{\mathsf{E}}^{(j)}[\mathbf{h}(\mathbf{x})]^{\mathrm{T}},$$
$$\hat{\mathsf{E}}^{(j)}[(\mathbf{x} - \hat{\mathbf{x}}^{(j)})(\mathbf{h}(\mathbf{x}) - \hat{\mathsf{E}}^{(j)}[\mathbf{h}(\mathbf{x})])^{\mathrm{T}}] = \hat{\mathsf{E}}^{(j)}[\mathbf{x}\mathbf{h}(\mathbf{x})^{\mathrm{T}}] - \hat{\mathbf{x}}^{(j)}\hat{\mathsf{E}}^{(j)}[\mathbf{h}(\mathbf{x})]^{\mathrm{T}}.$$

Instead of specification of a fixed total number of iterations, it is possible to monitor error of the means $\hat{\mathsf{E}}^{(j)}[\mathbf{h}(\mathbf{x})]$, $\hat{\mathsf{E}}^{(j)}[\mathbf{h}(\mathbf{x})\mathbf{h}(\mathbf{x})^{\mathrm{T}}]$ and $\hat{\mathsf{E}}^{(j)}[\mathbf{x}\mathbf{h}(\mathbf{x})^{\mathrm{T}}]$ estimates, compare the error with a pre-specified threshold, and iterate until error of all statistic estimates are lower than the threshold.

V. Numerical Illustration

Consider the following nonlinear non-Gaussian system [18] with one-dimensional state

$$x_{k+1} = \phi_1 x_k + 1 + \sin(\omega \pi k) + w_k \tag{27}$$

with the state noise w_k described by the Gamma pdf $Ga(3, 2)$, $\forall k$, $\phi_1 = 0.5$, $\omega = 0.04$ are scalar parameters and $k = 1, \ldots, 60$. The state is observed by the scalar measurement described by the equation

$$z_k = \begin{cases} \phi_2 x_k^2 + v_k, & k \le 30, \\ \phi_3 x_k - 2 + v_k, & k > 30. \end{cases} \tag{28}$$

The measurement z_k is influenced by the measurement noise $v_k \sim \mathcal{N}\{v_k; 0, 10^{-5}\}$, $\forall k$, and the scalar parameters are $\phi_2 = 0.2$ and $\phi_3 = 0.5$. The initial condition is given by a sum of five Gaussian pdf's $p(x_0) = \sum_{j=1}^{5} \tilde{\mathbf{w}}_{-1}^{(j)} \times \mathcal{N}\{(x_0; \hat{x}_0^{(j)}, P_0^{(j)}\} = \sum_{j=1}^{5} 0.2 \times \mathcal{N}\{x_0; j - 3, 10\}$. The predictive pdf $p(x_0|z^{-1})$ is equal to $p(x_0)$.

For the purposes of the Gaussian sum global filters, we calculated a three-term Gaussian sum approximation of $Ga(3, 2)$ distribution by means of the expectation maximization algorithm as

$$\tilde{p}(w_k) = 0.29 \times \mathcal{N}\{w_k; 2.14, 0.72\} + 0.18 \times \mathcal{N}\{w_k; 7.45, 8.05\} + 0.53 \times \mathcal{N}\{w_k; 4.31, 2.29\},\ \forall k.$$

Performance of the following state estimation methods was compared in the numerical example:

- *global filters:*
  - Gaussian sum filter with the RUT (GSF-RUT),
  - Gaussian sum filter with Taylor expansion of the nonlinear equations [4] (GSF-TE),
  - Gaussian sum filter with unscented transform [20] (GSF-UT).
- *local filters:*
  - EKF,
  - UKF,
  - RUKF.

Within all global filtering methods, the pruning step had to be implemented to prevent exponential increase of the number of terms in the filtering Gaussian sum pdf. More specifically, at each time instant, the 20 highest-weighted terms were kept while the other were discarded.

The GSF-RUT selected parameters were the maximum number of iterations $N_{max} = 500$ and the error tolerance $\varepsilon = 0.5$. The efficient implementation of the RUT given by Algorithm 3 was used. In this example, computational costs of the efficient implementation amounts to approximately 7% of the computational costs of the simplest implementation described in the previous section.

The experiments were carried out using $M = 1000$ MC simulations. Due to the global property of the estimates produced by the GSF-RUT, five metrics were chosen for comparison of the obtained results:

- *Root Mean-Square Error (RMSE)* defined as

  $$\mathrm{RMSE}_k = \sqrt{\frac{1}{M} \sum_{i=1}^{M} (\hat{x}_{k|k}(i) - x_k(i))^2},$$

  where $x_k(i)$ and $\hat{x}_k(i)$ denote true and estimated state at the i-th MC run.
  The RMSE metric provides an evaluation of the estimate error expressed as the Euclidean distance between the true state and its estimate. The value of the RMSE provides an absolute evaluation of the estimate error,

- *Log Mean-Square Error Ratio (log-MSER)* given by

$$\text{log-MSER}_k = \frac{1}{M}\sum_{i=1}^{M}\frac{\text{tr}(\boldsymbol{\Sigma}_k)}{\text{tr}(\mathbf{P}_{k|k}(i))},$$

where $\mathbf{P}_{k|k}(i)$ is the covariance matrix of the estimate provided by the filter at the i-th MC run and $\boldsymbol{\Sigma}_k$ is the mean-square error of the estimate. The log-MSER is a scalar measure of the normalized non-credibility *matrix* (NNCM), $NNCM = \mathbf{P}_{k|k}^{-1/2}\boldsymbol{\Sigma}_k\mathbf{P}_{k|k}^{-1/2} - \mathbf{I}$ [26]. According to the values of the log-MSER (positive/negative) the estimator is said to be optimistic or pessimistic.
- *Non-Credibility Index (NCI)* defined as [11], [27]

$$\begin{aligned}\text{NCI}_k =& \frac{1}{M}\sum_{i=1}^{M}\Big[10\log_{10}\Big((x_k(i)-\hat{x}_{k|k}(i))^2/P_{k|k}(i)\Big)\Big) \\ &- 10\log_{10}\Big((x_k(i)-\hat{x}_{k|k}(i))^2/\Sigma_k\Big)\Big].\end{aligned}$$

The NCI provides an evaluation of a relative estimation error [28] and, moreover, it evaluates a self-assessment provided by each filter in the form of the covariance matrix of the estimate error.
Note that NCI is a scalar measure dependent on the distribution of the estimation error, while the non-credibility matrix is a matrix measure depending only on the actual and calculated second moments of the estimation error [26].
- *Averaged Normalized Estimation Error Squared (ANEES)* [32] defined as

$$\text{ANEES}_k = \frac{1}{n_x M}\sum_{i=1}^{M}\Big((\mathbf{x}_k^i - \hat{\mathbf{x}}_k^i)^{\mathrm{T}}(\mathbf{P}_{k|k}^i)^{-1}(\mathbf{x}_k^i - \hat{\mathbf{x}}_k^i)\Big).$$

The ANEES credibility measure has a nice property of dimension normalization [27].
- *Inaccuracy* defined as

$$\text{K}_k(p_1, p_2) = \int p_1(\mathbf{x}_k|\mathbf{z}^k)\log\frac{1}{p_2(\mathbf{x}_k|\mathbf{z}^k)}\text{d}\mathbf{x},$$

where p_1 is the true filtering pdf and p_2 is the filtering pdf obtained by the filters. The true pdf was computed in the form of an empirical pdf using a particle filter [29] with 10^5 samples. The global filters produced the approximate filtering pdf in the Gaussian sum form (8). For the local filters, which produce only the filtering mean and covariance matrix, the filtering pdf was assumed to be Gaussian.
Considering the true pdf p_1 in the empirical pdf form, the inaccuracy [30] is a suitable measure of discrepancy between the pdf's p_1 and p_2. It equals to the Kullback-Leibler distance between p_1 and p_2 increased by the Shannon differential entropy of p_1 [31].

The RMSE values are given in Figure 2, the log-MSER values in Figure 3, the NCI values in Figure 4, the ANEES values in Figure 5, and the inaccuracy in Figure 6. Note that the inaccuracy values for the EKF and UKF were by several orders higher than the values for the other filters and, therefore, for clarity purposes they are omitted in the figure. The inaccuracy values for the RUKF ans GSF-RUT were very close; hence they are depicted on a separate figure (Figure 7). Note that the abrupt changes at $k = 30$ are caused by the fact that at this time instant the measurement function becomes linear (see (28)). As was expected, the RMSE results show that the GSF-RUT achieves the smallest error of the global filters and also that global estimators perform better than the local ones. In terms of the log-MSER and the NCI, the GSF-RUT provides the best results among the global filters. However, the lowest values of log-MSER and NCI were achieved by the RUKF, which is very surprising as it is a local filter. The inaccuracy then again confirmed the expectations that global estimates are closer to the true filtering pdf than the local estimates represented by a Gaussian distribution. Among the global estimates the results with the lowest inaccuracy belong to the GSF-RUT.

In summary, the GSF-RUT achieved the best results in terms of the density estimates. However, this fact does not necessarily imply that the moments calculated from the best

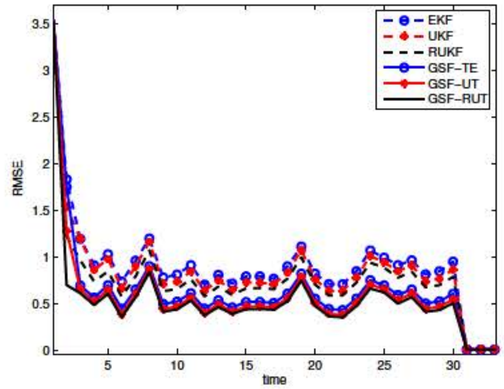

Figure 2. Time development of the RMSE.

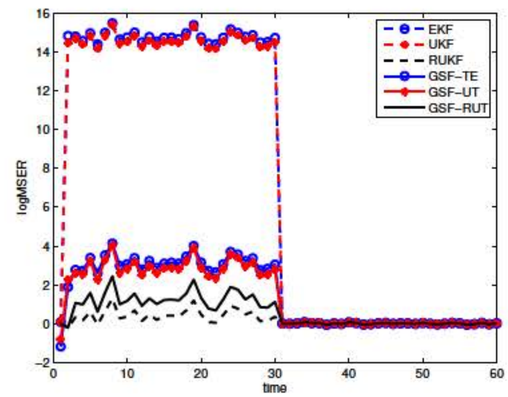

Figure 3. Time development of the log-MSER.

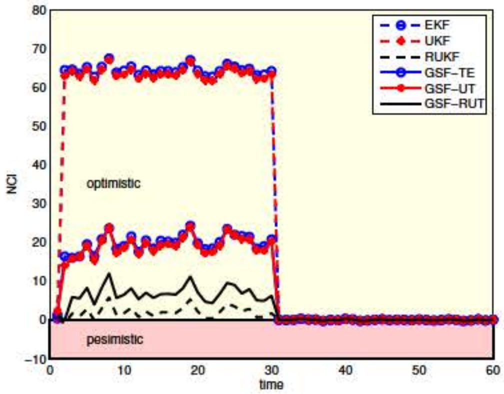

Figure 4. Time development of the NCI.

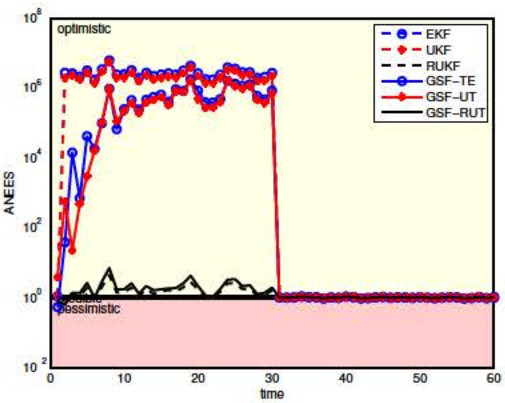

Figure 5. Time development of the ANEES.

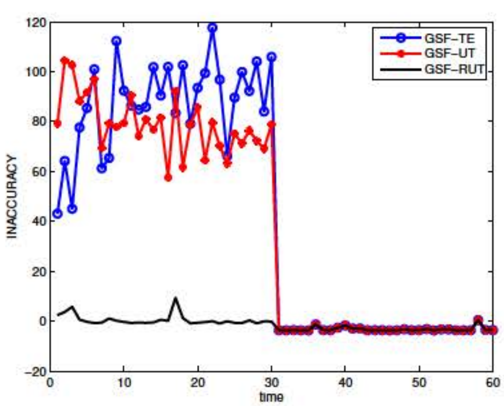

Figure 6. Time development of the inaccuracy.

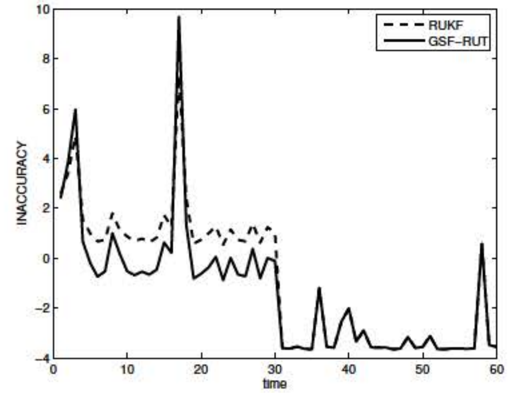

Figure 7. Time development of the inaccuracy for the RUKF and GSF-RUT (detail).

	EKF	UKF	RUKF	GSF-TE	GSF-UT	GSF-RUT
Time [msec]	0.4	0.5	3.1	5.2	7.9	7.1

Table I
COMPUTATIONAL COSTS OF A TIME STEP [MSEC]

filtering pdf estimate are also the best, as was illustrated by this example. Although the mean estimate achieved by the GSF-RUT was, according to the RMSE, the best, the RUKF accomplished estimate error variance closer to the true one than the GSF-RUT (measuring by the log-MSER). This could also lead to the claim that the RUKF is a more credible estimator than the GSF-RUT despite the fact that the GSF-RUT gives better density estimates. The reason for this situation may be the fact that the Gaussian sum based density estimate produced by the GSF-RUT is only an approximation of the true one. For the sake of completeness, the computational costs of a time step of each algorithm are given in Table I.

VI. CONCLUSION

The paper dealt with state estimation of nonlinear non-Gaussian systems by Gaussian sum filters. To achieve higher quality estimates, a Gaussian sum filter was proposed where the predictive state and measurement moments were computed by the randomized unscented transform (RUT). The RUT, which can be seen as a randomized version of the unscented transform, provides asymptotically exact estimates of the moments. To keep computational costs of the new filter low, an algorithm involving parallel calculation of the moments has been proposed which is termed the Gaussian Sum Filter with Randomized Unscented Transform (GSF-RUT). The GSF-RUT filter was illustrated in a numerical example and can compared to the local filters: extended Kalman Filter (EKF), Unscented KF (UKF), and randomized UKF (RUKF) as well the global filters of the Gaussian Sum Filter using the Taylor's expansion (GSF-TE) and unscented transform (GSF-UT). Comparative result were analyzed in detail using several performance measures evaluating accuracy of both point estimates and density estimates. Three conclusions from the comparative study were (1) in all cases, the RUT improved the results over the UT, (2) the RUKF local filter resulted in a better log mean square error and credibility index, and (3) the global GSF-RUT filter had the lowest absolute inaccuracy.

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